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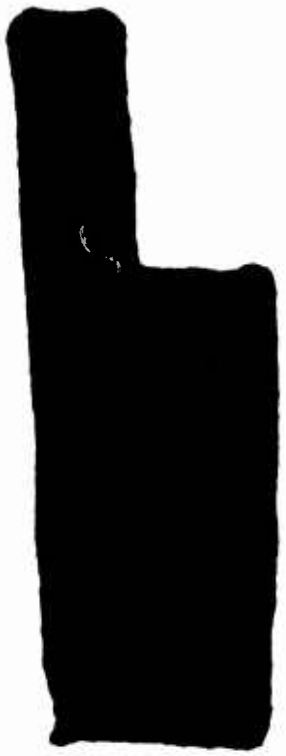

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COINCIDENCE COUNTING AND RENEWAL RUNS

by

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COINCIDENCE COUNTING AND RENEWAL RUNS

D. P. Gaver, Jr.

1. Introduction and Problem Description

In a Stanford technical report, [6], Zweig describes probability models for various aspects of the photographic detection process. In particular, a model is postulated for the process by which a chemical particle (silver halide grain) becomes a "developable speck" under the influence of photons acting upon it at random times. A brief account of the latter physical process is as follows: A particle may be in one of R + 1 distinct energy states, labelled, in order of increased energy, S_0, S_1, ... S_R; for simplicity these states will simply be denoted by the integers 0, 1, ... R. The particle's transitions between states are induced by light particles (photons)which impinge upon it; the time between arrival of consecutive photons is the random variable α. The interaction between the photons and the particle is such that if the particle is in the lowest energy state, 0, then a new photon causes it to jump to state 1. It remains there until either a temporary activation time, τ_1, elapses, or a new photon hits it, whichever event occurs first. If the temporary activation time terminates before another photon appears, the particle reverts to state 0. If the new photon arrives before τ_1 after the first photon, the particle jumps to state 2, a new temporary activation time, τ_2, is initiated, and the process begins as if from scratch, but starting from state 2. It is apparently in accord with physical understanding to

postulate that if a particle occupies any state up to and including R-1 then transitions will occur either to the next higher state, or all the way back to 0, depending upon the rapidity with which one photon follows the next. In contrast, state R is "absorbing", in the sense that a particle that reaches R remains there indefinitely. The physical interpretation of the latter event is that a particle in state R is a permanent "latent image speck", and becomes visible if exposed to development, i.e. chemical treatment. We study here the distribution of T_{OR}, the time until particle energy first reaches state R, starting from state 0. Since the activation of the particle -- the occurrence of a "count" -- is the result of coincidence of the photons, the process studied will be called coincidence counting.

Various probabilistic formulations of the latter problem are possible. That of [6] is to suppose that the photons appear according to a stationary Poisson process, and that each temporary activation time, τ, resulting after a new jump is an independent random variable possessing a density. Using these assumptions, [6] determines the transform of the distribution of the random time, T_{OR}, described above. This paper relaxes the assumptions of [6], replacing the assumption of Poisson-arriving photons by that of a general renewal process. The model of the present paper is an example of what Lévy and Smith have called semi-Markov processes; a comprehensive discussion of such processes has been given by

Pyke in [2] and [3], and our results can be derived by following the latter route. However, in order to keep the exposition self-contained and to show clearly all of the process components, we have chosen to argue directly and not to use the general theory.

Although the model outlined above was developed for a specific physical problem, it has possible interpretations in other fields. For example, it is tentatively suggested that the mechanism describes the process of learning certain skills. Suppose that a particular skill is only needed at random time intervals of duration α, and that the organism possesses a temporary skill retention time τ. This means that in general a skill is retained for a time τ following its last use, but that if the skill is not required again during τ it is forgotten and must be relearned. The exception is that if a critical number of needs for the skill develop in rapid succession, e.g. if R-1 uses are made of the skill, no two of which in succession are separated by more than τ, then the skill is permanent, and will not be forgotten. Thus the random variable T_{OR} represents the time until the skill becomes permanent.

It is interesting to compare the model outlined with the classical theory of runs for Bernoulli trials; see (Feller [1], p. 299 ff.). If we define a renewal run of length r to be an uninterrupted sequence of inter-event (photon arrival) times, each of which is shorter than τ, then

we are essentially interested in the distribution of the time until the termination of the first renewal run of r = R-1. Various of our results are similar to known results in run theory. Details are given in Sec. 3.

The present model may be modified to describe energy transitions down by only one step instead of all the way to zero. This is done in Sec. 4, assuming that photons arrive according to a stationary Poisson process.

2. A Probability Model

Let S(t) be the energy state of a particle at time t; S(t) assumes the values 0, 1, 2, ..., R. The random process $\{S(t),\ 0 \le t < \infty\}$ may be specified in terms of the time sequence (of photon arrivals) $\{t_i,\ i=1,2,\ldots\}$ as follows

$$S(t) = 0 \quad \text{for} \quad 0 \le t < t_1$$

$$S(t_1+) = 1 \tag{2.1}$$

$$S(t) = \begin{cases} S(t_i+) & \text{for} \quad t_i \le t < \min(t_i+\tau_i, t_{i+1}) \\ S(t_i+) + 1 & \text{for} \quad t_{i+1} \le t < \min(t_i+\tau_i, t_{i+2}); \ i=1,2,\ldots \\ 0 & \text{for} \quad t_i+\tau_i \le t < t_{i+1} \end{cases}$$

The arrival times t_i $(0 < t_1 < t_2 < \ldots)$ may be written as

$$t_i = \sum_{j=1}^{i} \alpha_j \qquad (2.2)$$

where $\{\alpha_j,\ j=1,2,\ldots\}$ is a renewal process: a sequence of independently distributed positive random variables, identically distributed according to the arbitrary distribution function (henceforth abbreviated d.f.) $F(\cdot)$; $F(0+) = 0$; α_j may be referred to as the j-th inter-event time. The moments of α_j will be assumed finite. To simplify the arguments we shall take all temporary activation times to be the same: $\tau_1 = \tau$, a constant independent of i. This assumption may be relaxed, and the results of doing so will be mentioned occasionally, under the hypothesis that $\{\tau_i\}$ is a sequence of independently and identically distributed positive random variables.

The primary purpose of the argument to follow is to characterize (by Laplace-Stieltjes transform) the d.f. of the fixation, or absorption, time, T_{OR}. The latter is the first-passage time from State 0 to State R:

$$T_{OR} = \min(t:\ S(0) = 0,\ S(t) = R)\ . \qquad (2.5)$$

It will then be possible to derive moments of the latter time, and, at least in principle, to exhibit the explicit d.f. of T_{OR}. In the course of the derivation to be given it is convenient, and possibly informative, to examine some auxiliary events and random variables. These will be introduced in what follows, and a final synthesis made to find the d.f. of T_{OR}.

(a) Pre-absorption energ. maxima

Prior to the time T_{OR}, at which absorption occurs, the particle may visit each of the intervening states 0, 1, ..., R-1 a number of times, each time dropping to 0 and starting afresh before finally reaching R. Following each return to 0 the particle eventually jumps to 1. Features of the particle's sojourn in States 1,2,...,R-1 will be treated here first.

Let $t(1;k)$ denote the photon arrival time immediately following which the particle is in state 1 for the k-th time. Thus, in terms of $\{t_i\}$, see (2.2),

$$t(1;1) = t_1 \tag{2.4}$$
$$t(1;k+1) = \min\{t_i: \quad t_i > t(1;k) \text{ and } S(t_i+) = 1\}.$$

Let $A_n(k)$ denote the event that the particle reaches state n, but not n+1, following $t(1;k)$. Then

$$p_n(k) = P[A_n(k)] = P[M(k) = n]; \tag{2.5}$$

where

$$M(k) = \max_{t(1;k)\leq t<t(1;k+1)} S(t) \tag{2.6}$$

It follows from (2.1) and the independence and identical distribution of the inter-event times that

$$p_n(k) = \prod_{j=1}^{n-1} P[\alpha_j(k) \le \tau]\, P[\alpha_j(k) > \tau) = [F(\tau)]^{n-1}[1-F(\tau)],\ n=1,\ldots,R-1$$

$$p_R(k) = \prod_{j=1}^{R-1} P[\alpha_j(k) \le \tau] = [F(\tau)]^{R-1} \tag{2.7}$$

where $\alpha_j(k)$ denotes the j-th inter-arrival time following $t(1;k)$.

If τ_i is a sequence of positive independent random variables with d.f. $G(\cdot)$, then (2.7) is replaced by

$$\begin{aligned} p_n(k) &= [F_G]^{n-1}[1-F_G] \qquad n = 1,2,\ldots,R-1 \\ p_R(k) &= [F_G]^{R-1} \end{aligned} \tag{2.8}$$

where

$$F_G = \int_0^\infty [1-G(x)]\,dF(x) \tag{2.9}$$

Since

$$\sum_{n=1}^{R} p_n(k) = 1, \tag{2.10}$$

absorption in state R occurs after $t(1;1)$, $t(1;2),\ldots t(1;k),\ldots$ Each time particle energy reaches state 1 an independent trial occurs -- one that culminates either in absorption in R, or in return to 0 and a new start after reaching state 1 again. Letting K denote the trial number on which absorption occurs, we have from the above that

$$\begin{aligned} P[K = j] &= (1-F^{R-1})^{j-1} F^{R-1}; \qquad j=1,2,\ldots; \\ E[K] &= F^{1-R} \end{aligned} \tag{2.11}$$

Finally, let $\bar{M}(k)$ be the conditional random variable which equals $M(k)$ given that the latter is not greater than R-1; $\bar{M}$ is a pre-absorption energy maximum. Clearly

$$P[\bar{M}(k)=n] = P[M(k)=n|M(k) \leq R-1] = P[M(k)=n|K>k]$$

$$= \frac{F^{n-1}(1-F)}{1-F^{R-1}} \qquad n=1,2,\ldots,R-1 \tag{2.12}$$

(b) Activation periods

It is useful to define the duration of an incomplete activation period:

$$I(k) = \inf\left\{t: \; S(t(1;k) + t) = 0, \; t < \infty\right\}$$
$$= \inf\left\{t: \; S(t(1;k) + t) = 0, \; k < K\right\}. \tag{2.13}$$

$I(k)$ is a random variable representing the period of time that elapses from an instant (the k-th such) immediately following which S = 1 until the energy state next drops to 0, having avoided absorption at R. In terms of our inter-event times, and temporary activation times,

$$I(k) = \sum_{j=1}^{\bar{M}(k)-1} \beta_j(k) + \tau \tag{2.14}$$

where $\beta_j(k)$ is the duration of the j-th inter-event time following $t(1;k)$, given that the inter-event time is shorter than the temporary activation

time τ; hence the sum is to be interpreted as zero if $\bar{M}(k) = 1$. We have

$$P[\beta_j \leq x] = \begin{cases} \frac{F(x)}{F(\tau)}, & 0 \leq x \leq \tau \\ 1, & \tau < x, \end{cases} \tag{2.15}$$

and

$$E[e^{-s\beta_j}] = \frac{\int_0^\tau e^{-sx} dF(x)}{F(\tau)} = \frac{\hat{F}(x;\tau)}{F(\tau)} . \tag{2.16}$$

We shall later require a random variable, γ, bearing relation to the inter-event time α in a manner complementary to β; γ_k is the <u>excess</u> of the duration of the first inter-event time following $t(1;k)$ which is longer than the corresponding temporary activation time. Its distribution is

$$P[\gamma_k \leq x] = \frac{F(x+\tau) - F(\tau)}{1-F(\tau)}, \quad 0 \leq x \tag{2.17}$$

and

$$E[e^{-s\gamma_k}] = \frac{e^{s\tau} \int_\tau^\infty e^{-sx} dF(x)}{1-F(\tau)} . \tag{2.18}$$

The Laplace-Stieltjes transform of the d.f. of $I(k)$ may be written down directly, using conditional expectations:

$$\begin{aligned} E[e^{-sI(k)} | \bar{M}(k)=n] &= \exp[-s\tau + s\sum_{j=1}^{n-1} \beta_j] \qquad n = 1,2,\ldots,R-1 \\ &= e^{-s\tau} \left[\frac{\hat{F}(s;\tau)}{F(\tau)}\right]^{n-1} . \end{aligned} \tag{2.19}$$

Now use (2.11) to remove the condition:

$$E[e^{-sI(k)}] = \sum_{n=1}^{R-1} p_n(k)\, E[e^{-sI(k)} | \bar{M}(k) = n] \tag{2.20}$$

$$= \sum_{n=1}^{R-1} e^{-s\tau} \left[\frac{\hat{F}(s;\tau)}{F}\right]^{n-1} \frac{F^{n-1}(1-F)}{1-F^{R-1}}.$$

$$= \frac{e^{-s\tau}(1-F)}{1-F^{R-1}} \left\{\frac{1-[\hat{F}(s;\tau)]^{R-1}}{1-F(s;\tau)}\right\} .$$

In the event that $\{\tau_j\}$ is a sequence of independent r.v, then

$$E[e^{-sI(k)}] = \frac{\int_0^\infty e^{-sx}[1-F(x)]dG(x)}{1-F_G^{R-1}} \left(\frac{1-\left\{\int_0^\infty e^{-sx}[1-G(x)]dF(x)\right\}^{R-1}}{1-\int_0^\infty e^{-sx}[1-G(x)]dF(x)}\right); \tag{2.21}$$

this latter expression follows by conditioning according to the values of τ_j as well as $\bar{M}$. It can be seen to reduce to (2.20) when $G(x)$ is degenerate, concentrating at τ.

A complete activation period, C, is defined as follows:

$$C = \inf\left\{t: \ S(t(1;K) + t) = R\right\} ; \tag{2.22}$$

C is the time required for the energy level to pass from state 1 to R without any intervening sojourns in state 0. Alternatively C may be expressed as

$$C = \sum_{j=1}^{R-1} \beta_j ; \tag{2.23}$$

it then follows immediately that

$$E[e^{-sC}] = \left[\frac{\hat{F}(s;\tau)}{F}\right]^{R-1} . \tag{2.24}$$

In the event that $\left\{\tau_i\right\}$ are distributed according to $G(\cdot)$,

$$E[e^{-sC}] = \frac{\left[\int_0^{\infty} e^{-sx}[1-G(x)]dF(x)\right]^{R-1}}{F_G} \tag{2.25}$$

(c) First-passage time, T_{OR}

Using the ingredients described above it is seen that the desired first-passage time may be expressed as

$$T_{OR} = \alpha_1 + \sum_{j=0}^{K-1} [\gamma_k + I(k)] + C \tag{2.26}$$

Given the value of K, T_{OR} is seen to be a sum of independent random variables, so

$$E[e^{-sT_{OR}}|K=k] = E(e^{-s\alpha})[E(e^{-s\gamma})E(e^{-sI})]^{k-1}\, e^{-sC}. \qquad (2.27)$$

When one removes the condition by use of (2.11):

$$E[e^{-sT_{OR}}] = \sum_{k=1}^{\infty} E[e^{-sT_{OR}}|K=k]\, P[K=k] \qquad (2.28)$$

the result is, after substituting and summing the geometric series,

$$E[e^{-sT_{OR}}] = \frac{E(e^{-s\alpha})\, E(e^{-sC})\, F^{R-1}}{1-E(e^{-s\gamma})E(e^{-sI})(1-F^{R-1})} \quad . \qquad (2.29)$$

If the previously obtained expressions for the component expectations are substituted and some simplification made,

$$E[e^{-sT_{OR}}] = \left\{ \int_0^{\infty} e^{-sx}\, dF(x) \right\} \cdot \frac{\left[\int_0^{\tau} e^{-sx}\, dF(x) \right]^{R-1}}{1 - \int_{\tau}^{\infty} e^{-sx}\, dF(x) \quad \left\{ \frac{1 - [\int_0^{\tau} e^{-sx} dF(x)]^{R-1}}{1 - \int_0^{\tau} e^{-sx}\, dF(x)} \right\}} \qquad (2.30)$$

3. Some Expected Values and a Limiting Distribution

The various component probability distributions and transforms derived in (a), (b), and culminating in (2.30) of (c) above, solve in principle the problem posed. In order to help the intuition in dealing with these expressions a few expectations will be presented.

Also , a limiting distribution for T_{OR} will be derived.

A. Moments

Expressions (2.5) and (2.8) give the probability distribution of the random variable M(k): the maximum state reached in a single (the k-th) sojourn in positive states. The generating function of M(k) is easily computed from (2.8), and from it

$$E(M(k)) = \frac{1-F_G^R}{1-F_G}; \tag{3.1}$$

clearly the latter approaches R as F_G increases, as it should.

Since the trial number, K, on which absorption occurs is again geometrically distributed, by (2.11), a generating function is immediately obtained; from this, or directly,

$$E[K] = F_G^{1-R} \tag{3.2}$$

Similarly, the expected single-trial pre-absorption energy maximum $\bar{M}(k)$, has a truncated geometric distribution, (2.12), and

$$E[\bar{M}(k)] = \frac{1 - R F_G^{R-1} + (R-1)F_G^R}{(1-F_G)(1-F_G^{R-1})} . \tag{3.3}$$

From (2.11) and (2.12) another piece of supplementary information is available: the distribution of highest state reached before absorption in state R. This is simply the d.f. of the unconditional pre-absorption energy maximum, denoted by $\bar{\bar{M}}_R$ when absorption level is R.

A conditional argument gives

$$P\left\{\bar{\bar{M}}_R \leq n\right\} = F_G^{R-1-n} \qquad n = 0,1,\ldots,R-1 \tag{3.4}$$

and the first moment is

$$E(\bar{\bar{M}}_R) = R - \frac{1-F_G^R}{1-F_G} \; . \tag{3.5}$$

Finally, differentiation of (2.30) gives the expected value of the first-passage time:

$$E(T_{OR}) = E(\alpha) \cdot \frac{F_G^{-R} - 1}{F_G^{-1} - 1} \tag{3.6}$$

B. Limiting Exponential Distribution for T_{OR}

It is apparent that $E(T_{OR})$ increases with R, and with decreases in F_G; for fixed F the latter behavior occurs if τ becomes small or if $G(\cdot)$ concentrates near zero. It is of interest to establish a limiting distribution for the scaled random variable

$$T^*_{OR} = T_{OR}\,[E(T_{OR})]^{-1}.$$

It will now be shown that under general conditions

$$P\left\{T_{OR} \leq t\, E(T_{OR})\right\} \longrightarrow 1 - e^{-t} \qquad \text{for} \qquad t \geq 0 \tag{3.7}$$

as $E(T_{OR}) \longrightarrow \infty$.

Verification of (3.7) follows from (2.29) and the continuity theorem for characteristic functions. The characteristic function of

$$T^*_{OR} = T_{OR}[E(T_{or})]^{-1}$$

is obtained from (2.29) by simply replacing s by

$$-i\xi[E(T_{OR})]^{-1}.$$

Now it is easy to see (for example, from (2.23), (2.14), and (2.15)) that the random variables C and I are with probability one, both bounded above by $(R-1)\tau$. From this it follows that

$$C[E(T_{or})]^{-1} \leq (R-1)\tau \frac{1-F(\tau)}{1-F^R(\tau)} F^{R-1}(\tau) \tag{3.8}$$

and, since we are assuming that $0 < F(\tau) < 1$ for $\tau > 0$, the right hand side tends to zero as either $R \longrightarrow \infty$ or $\tau \longrightarrow 0$, hence

$$E \exp(i\xi C[E(T_{OR})]^{-1}) \longrightarrow 1,$$

uniformly in any finite ξ-interval enclosing the origin. Similarly,

$$E \exp(i\xi\alpha[E(T_{OR})]^{-1}) \longrightarrow 1.$$

Next,

$$E(\exp i\xi\, \gamma[E(T_{OR})]^{-1}) = 1 + i\xi\, E(\gamma)[E(T_{OR})]^{-1} + r_1(\xi)$$

and (3.9)

$$E(\exp i\xi\, I[E(T_{OR})]^{-1}) = 1 + i\xi\, E(I)[E(T_{OR})]^{-1} + r_2(\xi),$$

where the remainders $r_1(\xi)$ and $r_2(\xi)$ are bounded in absolute value:

$$|r_1(\xi)| \leq \frac{\xi^2}{2} \frac{E(\gamma^2)}{[E(T_{OR})]^2} \leq \frac{\xi^2}{2} \frac{E(\alpha^2)}{E^2(\alpha)} [F(\tau)]^{2(R-1)} \frac{[1-F(\tau)]}{[1-F^R(\tau)]^2}$$

$$|r_2(\xi)| \leq \frac{\xi^2}{2} \frac{(R-1)^2\tau^2[F(\tau)]^{2(R-2)}[1-F(\tau)]^2}{E^2(\alpha) \quad [1-F^R(\tau)]^2} . \tag{3.10}$$

Note that $|r_1(\xi)|$ and $|r_2(\xi)|$ both tend uniformly to zero for all ξ in any finite interval enclosing the origin as either $R \longrightarrow \infty$ for fixed $\tau > 0$, or $\tau \longrightarrow 0$ for fixed $R \geq 2$. It then follows by expanding the denominator of (2.29) and using (3.9) and (3.10)

$$E \exp(i\xi \, T_{OR}[E(T_{OR})]^{-1}) \longrightarrow \frac{1}{1-i\xi} , \tag{3.11}$$

the characteristic function of the exponential distribution with unit mean, as $E(T_{OR}) \longrightarrow \infty$, where the latter limit is the result of either letting $R \longrightarrow \infty$ for τ fixed, or $\tau \longrightarrow 0$ for any fixed $R \geq 2$. By the continuity theorem for characteristic functions, (3.7) follows.

The above result suggests that if we modify the model so that when state R is reached the process state immediately reverts to zero, from which point the process starts over, then the distribution of the number of renewal runs of length R-1 accumulated in time $t' = t\,E(T_{OR})$ tends to the Poisson:

$$P\{N(t') = n\} = e^{-t} \frac{t^n}{n!} \qquad n = 0,1,2,... \tag{3.12}$$

as either $R \longrightarrow \infty$ or $\tau \longrightarrow 0$.

The strong resemblance between the present renewal run process and classical runs in Bernoulli trials has been mentioned earlier. Let the event that an inter-event time α is less than the corresponding temporary activation time τ be called a "success", and the complementary event a failure, so $F(\tau)$ or F_G is the probability of a "success". Then if the renewal run process is started from state 1 the expected time until a run of length r=R-1 occurs is, from (3.6),

$$E(T_{1R}) = E(\alpha) \cdot \frac{F_G^{-R}-1}{F_G^{-1}-1} - E(\alpha) = E(\alpha) \cdot \frac{1-F_G^r}{F_G^r[1-F_G]} , \qquad (3.13)$$

and if $E(\alpha) = 1$ this is the expected recurrence time of runs of length r, as given by Feller ([1], (7.7) p. 300). Similarly, if we consider the model, modified as described above by throwing away the initial time to go from 0 to 1, our previous argument shows that

$$P\left\{T_{1R} \leq t\, E(T_{1R})\right\} \longrightarrow 1 - e^{-t} \qquad (3.14)$$

as $R \longrightarrow \infty$, and consequently the distribution of the <u>number</u> of renewal runs in time $t' = t\, E(T_{1R})$ tends to the Poisson with unit mean. This result is entirely comparable to the Poisson distribution for long runs in Bernoulli trials; see Feller ([1], p. 310, problems 25 and 26).

4. A Modified Model

The model for energy state transitions specified by (2.1) permits state jumps either up by one unit, or down to state 0. As an alternative one may contemplate a process that permits transitions either up or down by one unit. Transition up by one unit is the result of a photon arrival; transition down by one unit occurs if no new arrival has occurred for a (random and independent) time period τ that begins immediately after the last transition, whether the latter was up or down; no down transitions are permitted from state 0, however.

Analysis of the process described is difficult for generally distributed inter-event times, α, because of the absence of convenient regenerative properties when down jumps occur. In our earlier model a down jump always terminated at zero. Analysis of the present model is rendered complex because the distribution of the excess occurring on the occasion of a downjump in general depends upon details such as the number of down jumps following the last arrival.

If, however, we consider the special case of stationary Poisson photon arrivals -- perhaps the most plausible physical assumption in any case -- the above difficulty vanishes because of the characteristic memoryless property of the Poisson. When the latter is in effect convenient regeneration points occur just following each state change. Let

$$F(x) = 1 - e^{-\lambda x} \qquad \lambda > 0 \qquad (4.1)$$

denote the exponential d.f. of α, a generic inter-event time. If $\{t_i', i = 1,2,\ldots\}$ is the sequence of times at which transitions -- either unit up or down jumps -- occur, then

$$p_k = P\{S(t_{i+1}'+) = k+1 | S(t_i'+) = k\} = \int_0^\infty [1-G(x)]e^{-\lambda x}\,\lambda dx$$
$$q_k = P\{S(t_{i+1}'+) = k-1 | S(t_i'+) = k\} = \int_0^\infty e^{-\lambda x}\,dG(x) \tag{4.2}$$

where $p_0 = 1$, $p_k+q_k = 1$. In words, p_k is the probability that a transition is an up jump, and q_k the probability that it is down. For the present process the probabilities p_k and q_k are independent of k, but this need not be true in general. We shall also be interested in the sojourn times of the process in the various states: an up sojourn time, $S_k^{(u)}$, is the time spent in state k, given that the next transition is to k+1; a down sojourn time, $S_k^{(d)}$, is the time spent in k, given that the next transition is to k-1. The distribution functions of these variables are

$$U_k(z) = P[S_k^{(u)} \le z] = P[t_{i+1}'-t_i' \le z | S(t_i'+) = k,\ S(t_{i+1}'+) = k+1]$$
$$D_k(z) = P[S_k^{(d)} \le z] = P[t_{i+1}'-t_i' \le z | S(t_i'+) = k,\ S(t_{i+1}'+) = k-1]. \tag{4.3}$$

In the present model

$$U_k(z) = \frac{\int_0^z [1-G(x)]e^{-\lambda x}\lambda dx}{\int_0^\infty [1-G(x)]e^{-\lambda x}\lambda dx} \tag{4.4}$$

$$D_k(z) = \frac{\int_0^\infty e^{-\lambda x}\, d\, G(x)}{\int_0^\infty e^{-\lambda x}\, d\, G(x)} \, .$$

A process of the type described may be called a semi-Markov birth and death process; cf. Pyke [2], [3] and Smith [5], for formal definition of the general S.-M.P. For present purposes it is sufficient to remark that the time to absorption in state R may be expressed as follows for the S.-M. birth and death process:

$$T_{OR} = A_0 + A_1 + \ldots + A_{R-1} \tag{4.5}$$

where $\{A_k\}$ is a sequence of independent random variables; A_k is the first-passage time from state k to state k+1:

$$A_k = \inf\{t: S(t_i'+t) = k+1, \text{ given } S(t_i'+) = k\} \, . \tag{4.6}$$

From the regenerative property of the process it is evident that A_k may usefully be expressed as

$$A_k = \sum_{n=0}^{J} [S_k^{(d)}(n) + A_{k-1}(n)] + S_k^{(u)}(J+1) \tag{4.7}$$

where $\{S_k^{(d)}(n)\}$ is the sequence of independent and identically distributed

sojourn durations in state k that terminate in transition down to k-1; $\{A_{k-1}(n)\}$ is the corresponding sequence of independently and identically distributed first-passage times up from k-1 to k; $S_k^{(u)}$ is the duration of the final sojourn in k leading to transition up to k+1; J is the random number of sojourns terminating in down transitions before the eventual jump up to J+1. Now if we condition on J we have by independence,

$$E[e^{-sA_k}|J] = \left\{E[e^{-sS_k^{(d)}}]\, E[e^{-sA_{k-1}}]\right\}^J E[e^{-sS_k^{(u)}}]$$

$$= \left\{\tilde{D}_k(s)\, E[e^{-sA_{k-1}}]\right\}^J \tilde{U}_k(s) \qquad (4.8)$$

where $\tilde{U}_k(s)$ and $\tilde{D}_k(s)$ represent the Laplace-Stieltjes transforms of $U_k(x)$ and $D_k(x)$. Since

$$P[J=j] = q_k^j\, p_k, \qquad (4.9)$$

removal of the condition on J leads to

$$E[e^{-sA_k}] = \frac{p_k\, \tilde{U}_k(s)}{1 - q_k\, \tilde{D}_k(s)\, E[e^{-sA_{k-1}}]} \,. \qquad (4.10)$$

Since there are no down jumps from k = 0, A_0 is simply the duration of the sojourn in state 0, presumed known. Hence in principle one may proceed by induction to find $E[e^{-sA_k}]$ explicitly. Finally, by independence

$$E[e^{-sT_{OR}}] = \prod_{k=0}^{R-1} E[e^{-sA_k}] \quad . \tag{4.11}$$

Observe that all of the above formulas specialize easily to simple random walks and birth-and death processes by appropriate choice of sojourn distributions. In the latter cases they provide convenient means for computing first-passage times in queues, etc.

The expression (4.10) may be differentiated bo generate moments of A_k; the moments of T_{OR} may similarly be expressed in terms of the latter by way of (4.11). We have in general

$$E[A_k] = \frac{E[S_k]}{p_k} + \frac{q_k}{p_k}\frac{E[S_{k-1}]}{p_{k-1}} + \frac{q_k q_{k-1}}{p_k p_{k-1}}\frac{E[S_{k-2}]}{p_{k-2}} + \cdots \frac{q_k q_{k-1}\cdots q_1}{p_k p_{k-1}\cdots p_1} E[S_0] \tag{4.12}$$

where

$$E[S_k] = p_k E[S_k^{(u)}] + q_k E[S_k^{(d)}] \quad . \tag{4.13}$$

Returning now to our physical model we note from (4.2) and (4.4) that p_k and $E[S_k]$ are independent of k. In fact

$$p_k = 1-\tilde{G}(\lambda), \quad q_k = \tilde{G}(\lambda) \qquad k=1,2,\ldots$$

$$E[S_k] = \frac{1-\tilde{G}(\lambda)}{\lambda} ,$$

$$p_0 = 1, \quad E[S_0] = \frac{1}{\lambda} \quad .$$

From (4.12) there results

$$E[A_0] = \lambda^{-1}$$

$$E[A_k] = \lambda^{-1} \left\{ \frac{1 - [\frac{\tilde{G}(\lambda)}{1-\tilde{G}(\lambda)}]^{k+1}}{1 - \frac{\tilde{G}(\lambda)}{1-\tilde{G}(\lambda)}} \right\}, \qquad (4.14)$$

and finally by summation

$$E[T_{OR}] = \lambda^{-1} \left[\frac{R\left\{1 - \frac{\tilde{G}(\lambda)}{1-\tilde{G}(\lambda)}\right\} - \frac{\tilde{G}(\lambda)}{1-\tilde{G}(\lambda)} + \left\{\frac{\tilde{G}(\lambda)}{1-\tilde{G}(\lambda)}\right\}^{R-1}}{1 - \left\{\frac{\tilde{G}(\lambda)}{1-\tilde{G}(\lambda)}\right\}^{2}} \right] \qquad (4.15)$$

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